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FIBRE BUNDLES AND THE EULER CHARACTERISTIC

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1. Introduction

For any fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ there are three important maps: the projection p, the fibre inclusion i, and the evaluation $\omega : \Omega B \to F$. In this paper we demonstrate formulas for each of these maps involving the Euler-Poincaré number of the fibre.

Let M be a compact topological manifold with possibly empty boundary $\dot{M}$, $\chi(M)$ the Euler-Poincaré number of M, G any space of homeomorphisms of M with a continuous action on M, $\omega : G \to M$ the evaluation map for some base point, $M \xrightarrow{i} E \xrightarrow{p} B$ any (locally trivial) fibre bundle, and $L \subset B$ a (possibly empty) subcomplex of the CW complex B.

Theorem A. *For connected M and any coefficients*

$$\chi(M)\omega^* = 0 : \tilde{H}^*(M) \to \tilde{H}^*(G) .$$

Theorem B. *There exists a transfer homomorphism $\tau : H^*(E, p^{-1}(L)) \to H^*(B, L)$ such that $\tau \circ p^* = \chi(M)1$ for any coefficients.*

Theorem C. *There exists a transfer homomorphism $\tau : H_*(B, L) \to H_*(E, p^{-1}(L))$ such that $p_* \circ \tau = \chi(M)1$ for any coefficients.*

Special cases of Theorem A were discovered by the author in [3] and [4]. Note that B and C reduce to the classical transfer theorem for covering spaces when M is a finite set of points. Borel proved a version of Theorem B for M a closed connected differentiable manifold and $M \xrightarrow{i} E \xrightarrow{p} B$ an "oriented" fibre bundle with structural group acting differentially on M and cohomology groups with fields of coefficients whose characteristics does not divide $\chi(M)$, [2]. This result was improved by the author in [1] and [3].

All these theorems are consequences of the next. Let $\dot{E}$ be the subspace of E consisting of those points of E which are in the boundaries of the fibres containing them. Then $(M, \dot{M}) \xrightarrow{i} (E, \dot{E}) \xrightarrow{p} B$ is a fibre pair. If $\dot{M}$ is empty, then $\dot{E}$ is empty.

Theorem D. *Let M^n be orientable and connected, and assume $\pi_1(B)$ acts*

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trivially on $H^n(M^n, \dot{M}; Z) \cong Z$. *Then there exists a* $\chi \in H^n(E, \dot{E}; Z)$ *such that* $i^*(\chi) = \chi(M)\mu$ *where* μ *generates* $H^n(M, \dot{M}; Z)$.

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2. Integration along the fibre

Here we record some well known facts concerning integration along the fibre. Suppose $(F, F') \to (E, E') \xrightarrow{p} B$ is a fibred pair, and L is a subcomplex of B. Then the Serre spectral sequence converges to $H^*(E, E' \cup p^{-1}(L); G))$ and $E_2^{p,q} \cong H^p(B, L; \{H^q(F, F'; G)\})$.

Suppose $\pi_1(B)$ operates trivially on $H^n(F, F'; Z) \cong Z$ and $H^i(F, F'; Z) \cong 0$ for $i > n$. Then integration along the fibre is defined as the composition

$$p_\natural : H^n(E, E' \cup p^{-1}(L)) \twoheadrightarrow E_\infty^{i-n,n} \rightarrowtail E_2^{i-n,n} \cong H^{i-n}(B, L; H^n(M, M'; G) \cong H^{i-n}(B, L; G) .$$

Integration along the fibre satisfies three properties:

a) If $E \xrightarrow{p} E' \xrightarrow{q} B$ are two fibrations, then

$$(q \circ p)_\natural = q_\natural \circ p_\natural .$$

b) If we have a fibre square

$$\begin{array}{ccc} (F, F') & \longrightarrow & (\bar{F}, \bar{F}') \\ \downarrow & & \downarrow \\ (E, E' \cup p^{-1}(L)) & \xrightarrow{\bar{f}} & (\bar{E}, \bar{E}' \cup \bar{p}^{-1}(\bar{L})) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ (B, L) & \xrightarrow{f} & (\bar{B}, \bar{L}) \end{array}$$

and (F, F') and $(\bar{F}, \bar{F}')$ both have cohomological dimension n, then

$$\begin{array}{ccc} H^i(E, E' \cup p^{-1}(L)) & \xleftarrow{\bar{f}^*} & H^i(\bar{E}, \bar{E} \cup p^{-1}(\bar{L})) \\ \downarrow{\scriptstyle p_\natural} & & \downarrow{\scriptstyle \bar{p}_\natural} \\ H^{i-n}(B, L; G) & \xleftarrow{\psi} & H^{i-n}(\bar{B}, \bar{L}; G) \end{array}$$

commutes, where ψ is induced by f^* and a homomorphism on the coefficient group corresponding to the map induced by $\bar{f} \mid (F, F')$.

c) If $u \in H^*(B, L; G)$ and $v \in H^*(E, E'; G')$ then $p_\natural(p^*(u) \cup v) = u \cup p'_\natural(v) \in H^*(B, L; G'')$, where G and G' pair to G'' and $p_\natural : H^*(E, E' \cup p^{-1}(L)) \to H^*(B, L)$, and $p'_\natural : H^*(E, E') \to H^*(B)$.

Dually, we may define $p^\natural$ as the composition

$$H_{i-n}(B, L; G) \cong E^2_{i-n,n} \twoheadrightarrow E^{\infty}_{i-n,n} \rightarrowtail H_i(E, E' \cup p^{-1}(L); G) .$$

Properties a) and b) hold in a dual formulation. For cap products

$$\cap : H^q(X, A_1; G) \otimes H_n(X, A_1 \cup A_2; G') \to H_{n-q}(X, A_2; G'')$$

we have the following formula:

$$p_*(a \cap p^{\natural}(y)) = p_{\natural}(\alpha) \cap y \in H_*(B, L; G'') ,$$

where $y \in H_*(B, L; G')$, $\alpha \in H^*(E, E'; G)$, $p^{\natural} : H_*(B, L) \to H_*(E, E' \cup p^{-1}(L))$, and $p_{\natural} : H^*(E, E') \to H^*(B)$.

3. Proof of Theorem D

Let G be a group of orientation-preserving homeomorphisms on M with compact-open topology acting transitively on $\mathring{M} = M - \dot{M}$. Let H be the subgroup of G leaving the base point $*$ fixed. We take $* \in \mathring{M}$.

Consider the universal principal bundle $G \to E_G \to B_G$. Then the classifying space for H is $B_H = E_G \times_G \mathring{M}$ since $G/H = \mathring{M}$. Let $\bar{B}_H$ denote $E_G \times_G M$, and let $\dot{B}_H$ denote $E_G \times_G \dot{M}$. We have the following diagram of fibre squares:

$$(1) \qquad \begin{array}{ccccc} M & \longrightarrow & M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ \pi^*(\bar{B}_H) & \xrightarrow{\tilde{j}} & \bar{\pi}^*(\bar{B}_H) & \longrightarrow & \bar{B}_H \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{\pi}} \\ B_H & \xrightarrow{j} & \bar{B}_H & \xrightarrow{\bar{\pi}} & B_G \end{array} \qquad \pi = \bar{\pi} \circ j : B_H \to B_G$$

Here j and $\tilde{j}$ are inclusion maps.

Lemma 1. *Regarding $\tilde{j}$ as a map of pairs*

$$\tilde{j} : (\pi^*(\bar{B}_H), \pi^*(\dot{B}_H)) \to (\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) .$$

Then j and $\tilde{j}$ are homotopy equivalences.

Lemma 2. $(\pi^*(\bar{B}_H), \pi^*(\dot{B}_H)) = (E_G \times_H M, E_G \times_H \dot{M})$.

Proof.

$$\begin{array}{ccc} M & & M \\ \downarrow & & \downarrow \\ E_G \times_H M & \longrightarrow & E_G \times_G M = \bar{B}_H \\ \downarrow & & \downarrow \\ E_G \times_G \mathring{M} = E_G/H = B_H & \xrightarrow{\pi} & B_G = E_G/G . \end{array}$$

The existence of this fibre square implies that $E_G \times_H M = \pi^*(\bar{B}_H)$.

Since M is oriented, $Z \cong H^n(M, M - *) \xrightarrow{i^*} H^n(M, \dot{M})$ is an isomorphism where i is inclusion. Thus by Lemmas 1 and 2 and the naturality of integration along the fibre (§ 2(b)) we have the following commutative diagram:

$$(2)\qquad \begin{array}{ccccc} H^n(E_G \times_H M, E_G \times_H (M - *)) & \xrightarrow{\bar{i}^*} & H^n(\pi^*(\bar{B}_H), \pi^*(\dot{B}_H)) & \underset{\cong}{\xleftarrow{\bar{j}^*}} & H^n(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \\ \cong\big\downarrow p_{\natural} & & \big\downarrow \bar{p}_{\natural} & & \big\downarrow \bar{\pi}_{\natural} \\ H^0(B_H) & \underset{\cong}{\xrightarrow{i^*}} & H^0(B_H) & \underset{\cong}{\xleftarrow{j^*}} & H^0(\bar{B}_H) \end{array}$$

Note that $p_{\natural}$ is an isomorphism because the fibre of the fibre pair $(E_G \times_H M, E_G \times_H (M - *)) \xrightarrow{p} B_H$ is $(M, M - *)$ which has the cohomology of $(R^n, R^n - 0)$; thus the spectral sequence for p takes a very simple form, and $p_{\natural}$ may be thought of as the Thom isomorphism.

Now we define $\underline{U} \in H^n(E_G \times_H M, E_G \times_H (M - *))$ by the equation $p_{\natural}(\underline{U}) = 1$. Define $\underline{U}_1 \in H^n(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H))$ by $\underline{U}_1 = (\bar{j}^*)^{-1}\bar{i}^*(\underline{U})$. Then $\bar{\pi}_{\natural}(\underline{U}_1) = 1 \in H^0(\bar{B}_H)$ by diagram (2).

We have the fibre square

$$(3)\qquad \begin{array}{ccc} (M, \dot{M}) & \longrightarrow & (M, \dot{M}) \\ \big\downarrow & & \big\downarrow \\ M \times (M, \dot{M}) & \longrightarrow & (\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{i} & \bar{B}_H \end{array}$$

arising from the fibre inclusion $M \xrightarrow{i} \bar{B}_H \to B_G$, and restricting diagram (2) to the bundles over the fibres yields

$$(4)\qquad \begin{array}{ccccc} H^n(\mathring{M} \times M, \mathring{M} \times M - \Delta) & \xrightarrow{1 \times i^*} & H^n(\mathring{M} \times M, \mathring{M} \times \dot{M}) & \xleftarrow{\bar{j}^*} & H^n(M \times M, M \times \dot{M}) \\ \cong\big\downarrow p_{\natural} & & \big\downarrow p_{\natural} & & \big\downarrow \bar{\pi}_{\natural} \\ H^0(\mathring{M}) & \underset{\cong}{\longrightarrow} & H^0(M^0) & \underset{\cong}{\longrightarrow} & H^0(M) \end{array}$$

where Δ denotes the diagonal.

Define $U \in H^n(\mathring{M} \times M, M \times M - \Delta)$ by $p_{\natural}(U) = 1$, and define $U_1 \in H^n(M \times M, M \times \dot{M})$ as image of U.

Now let $T : X \times Y \to Y \times X$ stand for the twisting map. Noting that

$T: \bar{\pi}^*(\bar{B}_H) \to \bar{\pi}^*(\bar{B}_H)$ arises from the restriction of the twisting map to $\bar{\pi}^*(\bar{B}_H) \subset \bar{B}_H \times \bar{B}_H$, we have a commutative diagram:

$$(5) \qquad \begin{array}{ccc} (\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) & \xrightarrow{T} & (\bar{\pi}^*(\bar{B}_H), T(\bar{\pi}^*(\dot{B}_H)) \\ \uparrow \tilde{i} & & \uparrow \\ (M \times M, M \times \dot{M}) & \xrightarrow{T} & (M \times M, \dot{M} \times M) \end{array}$$

where $\tilde{i}$ comes from the fibre square (3).

Define $\underline{U}_2 \in H^n(\bar{\pi}^*(\bar{B}_H), T(\bar{\pi}^*(\dot{B}_H))$ by $\underline{U}_2 = (-1)^n T^*(\underline{U}_1)$. Similarly define $U_2 \in H^n(M \times M, \dot{M} \times M)$. Then the naturality of integration along the fibre and diagram (5) implies that $\underline{U}$, $\underline{U}_1$ and $\underline{U}_2$ defined in the universal case pull back under inclusion to U, U_1 and U_2 defined in the product case.

Now consider $U_1 \cup U_2 \in H^{2n}((M, \dot{M}) \times (M, \dot{M}))$. We have a relative fibre bundle pair

$$(M, \dot{M}) \to (M \times M, (M \times \dot{M}) \cup (\dot{M} \times M)) \xrightarrow{\pi} (M, M) ,$$

and we may define integration along the fibre $\pi_\natural : H^i((M, \dot{M}) \times (M, \dot{M})) \to H^{i-n}(M, \dot{M})$. In this simple situation, $\pi_\natural$ is the same as the slant product with the fundamental class $z \in H_n(M, \dot{M})$ (that is, $\pi_\natural(y) = y/z$). We call $\chi = \pi_\natural(U_1 \cup U_2)$ the Euler class in $H^n(M, \dot{M})$. This definition is easily seen to agree with that of Spanier [5, p. 347]. Thus we have $\chi = \chi(M)\mu \in H^n(M, \dot{M})$ where μ is the appropriately chosen generator.

On the other hand we have

$$\underline{U}_1 \cup \underline{U}_2 \in H^{2n}(\bar{\pi}^*(\dot{B}_H), \bar{\pi}^*(\bar{B}_H) \cup T(\bar{\pi}^*(\dot{B}_H))) .$$

Note that $T(\bar{\pi}^*(\dot{B}_H)) = \bar{\pi}^{-1}(\dot{B}_H)$. Thus we are lead to consider the relative fibre bundle pair

$$(M, \dot{M}) \to (\bar{\pi}^*(\dot{B}_H), \bar{\pi}^*(\bar{B}_H) \cup \bar{\pi}^{-1}(\dot{B}_H)) \xrightarrow{\pi} (B_H, \dot{B}_H) .$$

Thus we have integration along the fibre

$$\pi_\natural : H^i(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H) \cup \bar{\pi}^{-1}(\dot{B}_H)) \to H^{i-n}(\bar{B}_H, \dot{B}_H) .$$

Define the Euler class $\chi = \pi_\natural(U_1 \cup U_2) \in H^n(\bar{B}_H, \dot{B}_H)$. By naturality of $\pi_\natural$, we see that $i^*(\chi) = \chi(M)\mu$ for $i : (M, \dot{M}) \to (\bar{B}_H, \dot{B}_H)$, the fibre inclusion.

Since $(M, \dot{M}) \to (\bar{B}_H, \dot{B}_H) \xrightarrow{\bar{\pi}} B_G$ is the universal bundle pair for bundle pairs $(M, \dot{M}) \to (E, \dot{E}) \to B$ with structural group preserving the orientation of $(M, \dot{M})$, we always can find a fibre square

$$(6)\qquad \begin{array}{ccc} (M, \dot{M}) & \xrightarrow{1} & (M, \dot{M}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ (E, \dot{E}) & \xrightarrow{\bar{f}} & (\bar{B}_H, \dot{B}_H) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{f} & B_G \,. \end{array}$$

Define $\chi \in H^n(E, \dot{E})$ by $\chi = \bar{f}^*(\chi)$. It is clear that $i^*(\chi) = \chi(M)\mu$, so Theorem D is proved.

Note that every possible $\bar{f}$ which arises in diagram (6) must be fibrewise homotopic to any other [4], so χ is uniquely defined.

4. Proof of Theorem A

It is clear that Theorem A would be true in general if we can prove Theorem A for the case where G is the identity component of the group of homeomorphisms of M. So we make that assumption.

First we shall prove Theorem A when M is an oriented manifold. We have the fibre square

$$(7)\qquad \begin{array}{ccc} G \times (M, \dot{M}) & \xrightarrow{\hat{\omega}} & (M, \dot{M}) \\ \downarrow{\scriptstyle i \times 1} & & \downarrow \\ E_G \times (M, \dot{M}) & \xrightarrow{\phi} & (E_G \times_G M, E_G \times_G \dot{M}) \\ \downarrow & & \downarrow \\ B_G & \longrightarrow & B_G \end{array}$$

where $\hat{\omega}$ is the action of G on M, and ϕ takes $(e, x) \mapsto \langle e, x \rangle$. Since G is connected, we may apply Theorem D to the fibration on the right. Thus $\hat{\omega}^*(\chi(M)\mu)$ $(i \times 1)^*\phi^*(\chi)$. Since E_G is contractible, we see that

$$\hat{\omega}^*(\chi(M)\mu) = 1 \times (\chi(M)\mu) \in H^n(G \times (M, \dot{M}); Z) \,.$$

Let $\alpha \in H^i(M; G)$ be any element for $i > 0$. Then $\alpha \cup (\chi(M)\mu) \in H^{n+i}(M, \dot{M}; G)$ $\cong 0$. Thus

$$\begin{aligned} 0 &= \hat{\omega}^*(\alpha \cup (\chi(M)\mu) = \hat{\omega}^*(\alpha) \cup (\hat{\omega}^*(\chi(M)\mu) \\ &= ((\omega^*(\alpha) \times 1) + \text{other terms}) \cup (1 \times (\chi(M)\mu) \\ &= (\omega^*(\alpha) \times (\chi(M)\mu)) + (\text{other terms}) \cup (1 \times \chi(M)\mu) \\ &= \omega^*(\alpha) \times (\chi(M)\mu) = \chi(M)\omega^*(\alpha) \times \mu \,. \end{aligned}$$

Hence $\chi(M)\omega^*(\alpha) = 0$ when M is oriented.

Now we assume that M is unoriented. Let $\tilde{M}$ be the oriented double covering of M, and D the mapping cylinder of the projection $\tilde{M} \to M$. Then D is a manifold with boundary. We may think of G as acting on $\tilde{M}$ by lifting every homeomorphism $h : \mathrm{M} \to M$ to that lifting $\tilde{h} : \tilde{M} \to \tilde{M}$ which preserves orientation. Then G acts on D as a group of homeomorphisms by $g(x, t) = (\tilde{g}(t), t)$.

Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \omega} & \searrow^{\omega} & \\ M & \xrightarrow{i} & D \end{array} \tag{8}$$

Since the inclusion i is a homotopy equivalence, Theorem A holds for $G \xrightarrow{\omega} M$ if it holds for $G \xrightarrow{\omega} D$. But this is the case as follows from the following lemma.

Lemma 3. *D is orientable, and G preserves the orientation.*

Proof. First assume that M is closed. Then $\dot{D} = \tilde{M}$ and is orientable. An examination of the homology exact sequence of the pair $(D, \dot{D})$ shows that $H_{n+1}(D, \dot{D}) \cong Z$. So D is orientable.

Now assume that M has nonempty boundary $\dot{M}$. Then $\dot{D} = \tilde{M} \cup D(\dot{M})$ where $D(\dot{M})$ is the mapping cylinder of $\tilde{M} \xrightarrow{p} M$ restricted to $\partial\tilde{M} \to \dot{M}$. Now either $D(\dot{M})$ is $\dot{M} \times I$ in case $\dot{M}$ is orientable or it is the mapping cylinder of the bundle covering of $\dot{M}$. In either case $D(\dot{M})$ is orientable. Thus $\dot{D}$ is orientable. Then the homology exact sequence of $(D, \dot{D})$ implies that D is orientable. It is easily seen that G preserves the orientation.

5. Proof of Theorem B

We first proves Theorem B for the case when M is connected and orientable and $\pi_1(B)$ operates trivially on $H^n(M^n, \dot{M}) \cong Z$ in the fibration $(M, \dot{M}) \to (E, \dot{E}) \xrightarrow{\pi} B$.

Define $\tau : H^*(E, p^{-1}(L); G) \to H^*(B; G)$ by letting $\tau(\alpha) = \pi_!(\alpha \cup \chi)$.

Lemma 4. *$\tau \circ p^*(\alpha) = \chi(M)\alpha$ for all $\alpha \in H^*(B, L; G)$.*

Proof. From the fibre square

$$\begin{array}{ccc} (M, \dot{M}) & \xrightarrow{1} & (M, \dot{M}) \\ \downarrow{\scriptstyle 1} & & \downarrow \\ (M, \dot{M}) & \xrightarrow{i} & (E, \dot{E}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ * & \xrightarrow{c} & B \end{array} \tag{9}$$

we have $\pi_\natural(\chi) = \pi'_\natural i^*(\chi)$ by identifying $H^0(*)$ with $H^0(B)$. So $\pi_\natural(\chi) = \pi'_\natural(i^*(\chi)) = \pi'_\natural(\chi(M)\mu) = \chi(M)\pi'_\natural(\mu) = \chi(M)1$. Hence $\tau \circ p^*(\alpha) = \pi_\natural(p^*(\alpha) \cup \chi) = \alpha \cup \pi_\natural(\chi) = \alpha \cup (\chi(M)1) = \chi(M)\alpha$.

From now on we shall surpress L and $p^{-1}(L)$ in our notation.

Next we shall show Theorem B is true for M unoriented and connected. Let D be the mapping cylinder as in diagram (8). The projection $q : D \to M$ is equivariant with respect to the action of G. Thus we get a fibre square

$$\text{(10)} \qquad \begin{array}{ccc} D & \xrightarrow{q} & M \\ \downarrow & & \downarrow \\ \bar{E} & \xrightarrow{\tilde{q}} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{1} & B . \end{array}$$

The left fibration satisfies the previous case since D is oriented and G preserves the orientation by Lemma 3, so there exists a transfer $\tau_1 : H^*(\bar{E}; G) \to H^*(B; G)$. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1\tilde{q}^*$. Then $\tau \circ p^* = \tau_1\tilde{q}^*p^* = \tau_1 p_1^* = \chi(D)1 = \chi(M)1$.

Now we assume that M is orientable and connected but that $\pi_1(B)$ does not act trivially on $H^n(M, \dot{M}; Z)$. Then we obtain the commutative diagram

$$\begin{array}{ccc} M \times P^2 & \xrightarrow{\pi} & M \\ \downarrow & & \downarrow \\ E \times P^2 & \xrightarrow{\pi} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

where P^2 is the real projective plane, and π is projection on the first factor. The fibre bundle on the left satisfies the above case since $M \times P^2$ is unorientable. Thus there exists a transfer $\tau_1 : H^*(E \times P^2; G) \to H^*(B; G)$. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1\pi^*$. Then $\tau \circ p^* = \tau_1\pi^*p^* = \tau_1 p_1^* = \chi(M \times P^2)1 = \chi(M)1$.

Now assume that M is not connected. Then the fibre bundle $M \to E \xrightarrow{p} B$ factors through the fibre bundles $E \xrightarrow{p_2} \tilde{B} \xrightarrow{p_1} B$, where $\tilde{B}$ is an N-fold covering of B, and M is N disjoint copies of M_0. Thus we have a transfer for $M_0 \to E \xrightarrow{p_2} \tilde{B}$; call it τ_2. Also we have the classical transfer for the covering τ_1. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1 \circ \tau_2$. Then $\tau \circ p^* = \tau_1 \circ \tau_2 \circ p_2^* \circ p_1^*$

$= \tau_1 \circ \chi(M_0)1 \circ p_1^* = \chi(M_0)\tau_1 \circ p_1^* = N\chi(M_0)1 = \chi(M)1.$

In the case where E is not connected, we obtain a transfer for each component of E. Then we sum them to obtain the transfer for $E \xrightarrow{p} B$. Finally, if B is not connected, (we assume that each fibre of $E \xrightarrow{p} B$ is M), then the direct sum of the transfers over each component of B will yield the transfer we seek.

6. Proof of Theorem C and remarks

We begin as before, by assuming that E and M are connected and M is orientable, and that $\pi_1(B)$ preserves orientation. Then we have the Euler class $\chi \in H^n(E, \dot{E})$. Define the transfer $\tau : H_*(B, L; G) \to H_*(E, \pi^{-1}(L); G)$ by $\tau(\alpha) = \chi \cap \pi^\natural(\alpha)$ where $\pi^\natural : H^*(B, L; G) \to H^*(E, \dot{E} \cup \pi^{-1}(L); G)$. Then $p_* \circ \tau(\alpha) = p_*(\chi \cap \pi^\natural(\alpha)) = \pi_\natural(\chi) \cap \alpha = \chi(M)1 \cap \alpha = \chi(M)\alpha$.
The remainder of the proof is dual to § 5.

Several remarks are in order.

1. Various other transfers may be defined based on characteristic numbers of a manifold, however, not in the generality as the one we have defined. The essential point is to find the appropriate version of Theorem D. For example, if M^n is a closed connected differential manifold, $M \xrightarrow{i} E \xrightarrow{p} B$ is a fibre bundle with structural group G acting differentially on M, and M has a non-zero Pontryagin number p_I, then there is a class $\nu \in H^n(E; Z)$ such that $i^*(\nu) = p_I\mu$. Then we may prove, as before, that $p_I\omega^* = 0$ where $\omega : G \to M$ is the evaluation map from the structural group G, and obtain transfer theorems but only under the above restricted hypothesis. To see that $p_I\mu$ is in the image of i^*, we follow the idea of Borel [2, Lemma 3.2]. Similarly, for M a closed connected topological manifold we may define transfers (in Z_2 coefficients) by using Stiefel-Whitney numbers.

2. Theorem D is true for Z_2 coefficients with no orientability condition on M or the fibre bundle.

3. The Euler-Poincaré number in Theorems A, B and C is essential. For example, $O(3)$ acts on S^2 and it is well known that $\omega^* : \tilde{H}^*(S^2; Z_2) \to \tilde{H}^*(O(3); Z_2)$ is not that trivial homomorphism. But $\chi(S^2)\omega^* = 2\omega^* = 0$ since 2 is zero in Z_2. An example in the case of the transfer comes from the universal principal bundle $G \xrightarrow{i} E_G \xrightarrow{p} B_G$. Here $\tau \circ p^* = \chi(G)1$. But $\tilde{H}^*(E_G) = 0$. So $\chi(G) = 0$.

4. Applications will appear elsewhere. Among them they include the fact that RP^{2n} or CP^{2n} or QP^{2n} or *Cayley* p^2 do not fibre with a manifold as a fibre.

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